Asymptotic behaviour of heavy-tailed branching processes in random environments

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(Joint work with Professor Wenming Hong)

The 15th Workshop on Markov Processes and Related Topics

July 11-15, 2019

Overview

1. Martingale convergence: Branching processes
   - Kesten-Stigum Theorem
   - Seneta-Heyde theorem
   - $m = \infty$
2. Branching processes in random environments
   - Kesten-Stigum type Theorem
   - Seneta-Heyde type Theorem
   - Main results: Heavy-tailed $d(\bar{\xi}, s) = 0$

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Branching processes

$\{Z_n\}$ be a Galton-Watson branching process

- $Z_0 = 1$;
- $Z_n := \sum_{k=1}^{Z_{k-1}} \zeta_k$;
- ζ_k i.i.d., with $f(s) = \sum_{j=0}^{\infty} p_j s^j$;
- $m := EZ_1 = f'(1)$;
- $q := P\{Z_n \to 0,\ n \to \infty\}$;
- $m > 1$(supercritical case): $P(Z_n \to \infty) = 1 - q > 0$.
- question: $Z_n \xrightarrow{?} \infty$

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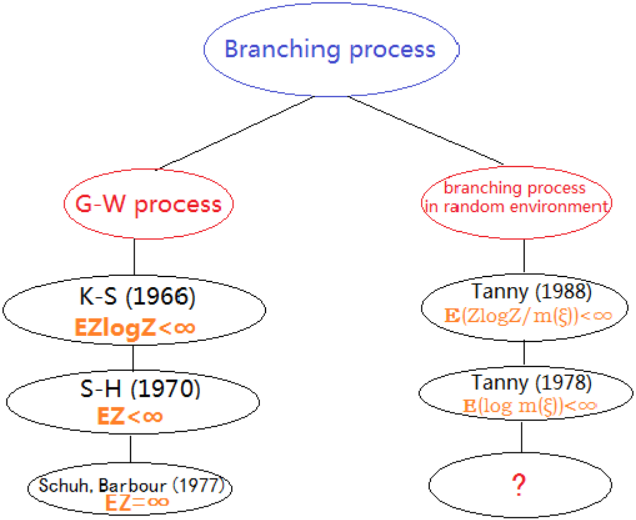

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EZlogZ<∞
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Kesten-Stigum Theorem

$W_n := \{\frac{Z_n}{m^n}\}$ is a martingale $\Rightarrow W_n :\to W \;\; a.s.$

Theorem (Kesten-Stigum, 1966)

$$EW = 1 \Leftrightarrow EZ_1 \log Z_1 < \infty.$$

Remark

$EW = 1 \iff W$ *is proper, i.e.,* $P(0 < W < \infty) = 1 - q.$

If only $EZ_1 < \infty$?

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Seneta-Heyde Theorem

- $f_n(s)$ to denote the probability generating function of Z_n;
- $k_n(s) = -\log f_n(e^{-s})$; $s \in (0, -\log q)$
- $h_n(s) = k_n^{-1}(s)$;

Theorem (Seneta (69), Heyde (70))

$X_n(s) := \exp(-Z_n h_n(s))$ is a martingale, $s \in (0, -\log q)$

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$W(s)$ is proper (i.e., $\mathbb{P}(0 < W(s) < \infty) = 1 - q$) if $EZ_1 < \infty$.

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- Seneta (1969), showed that it is never possible to find $\{c_n\}$ such that $\{\frac{Z_n}{c_n}\}$ converges in distribution to a proper, non-degenerate law.
- Darling (70) and Seneta (73) gave sufficient conditions for the existence of a sequence $\{c_n\}$ such that $\{\frac{\log(Z_n+1)}{c_n}\}$ converges in distribution to a non-degenerate law.

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- classification: regular or irregular, according to the property that whether there exists a sequence of constants $\{c_n\}$ such that $P(0 < \lim_{n\to\infty} \frac{Z_n}{c_n} < \infty) > 0$;
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Overview

Branching processes in random environments

- Environment: $\bar{\xi} = \{\xi_n : n \in \mathbb{Z}\}$ i.i.d.;
- $$\xi_n = \{\xi_n^{(0)}, \xi_n^{(1)}, \cdots\}, \qquad \xi_n^{(i)} \geqslant 0, \qquad \sum_{i=0}^{\infty} \xi_n^{(i)} = 1.$$

The law of the environment $\bar{\xi}$ is given by η.

- quenched law: $P_{\bar{\xi}}$;
- annealed law: $\mathbb{P}(\cdot) := \int P_{\bar{\xi}}(\cdot)\eta(d\bar{\xi})$.

Some notations:

- $m(\xi_0) = E_{\xi_0}(Z_1) := \sum_{y=0}^{\infty} y\xi_0^{(y)}$;
- $k_{\xi_i}(s) = -\log f_{\xi_i}(e^{-s})$; $\quad h_{\xi_i}(s) = -\log f_{\xi_i}^{(-1)}(e^{-s})$, $0 < s < \infty$;
- $k_n(\bar{\xi}, s) := k_{\xi_0}(k_{\xi_1}(\cdots(k_{\xi_{n-1}}(s))\cdots) =$
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Theorem (Tanny, 88)

$w.p.1,$

$$\lim_{n\to\infty} \frac{Z_n}{\pi_n} = W$$

$and\ \mathbb{P}(0 < W < \infty|\bar{\xi}) = 1 - q(\bar{\xi}) \Longleftrightarrow \mathbb{E}(Z_1 \log^+ Z_1/m(\xi_0)) < \infty.$

If only $\mathbb{E}|\log m(\xi_0)| < \infty$?

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and $\mathbb{P}(0 < W < \infty|\bar{\xi}) = 1 - q(\bar{\xi}) \iff \mathbb{E}(Z_1 \log^+ Z_1/m(\xi_0)) < \infty.$

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$\mathbb{E}|\log m(\xi_0)| < \infty \implies W(\bar{\xi}, s)$ *is proper*
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Question: when $\mathbb{E}|\log m(\xi_0)| = \infty$?

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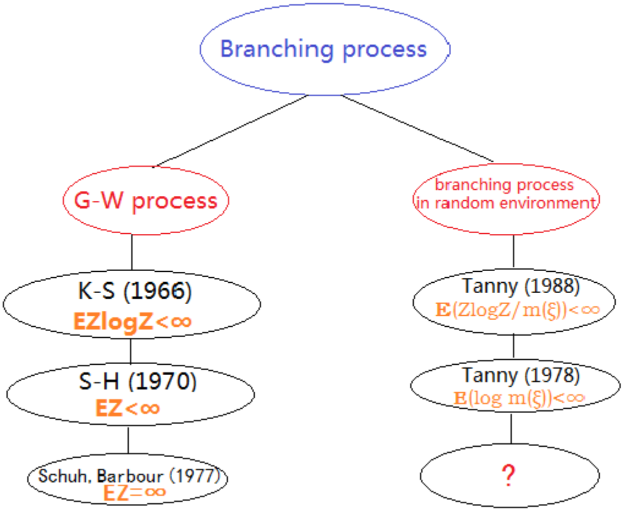

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- Let $W_n(\bar{\xi}, s) = Z_n(\bar{\xi}) h_n(\bar{\xi}, s), \quad X_n(\bar{\xi}, s)^u = e^{-uW_n(\bar{\xi}, s)}$,
- $d(\bar{\xi}, s) := \lim_{n \to \infty} \frac{h_{n+1}(\bar{\xi}, s)}{h_n(\theta\bar{\xi}, s)}$.
- key step $\mathbb{E}|\log m(\xi_0)| < \infty \Longrightarrow 0 < d(\bar{\xi}, s) \leqslant 1 \quad w.p.1.$
- Let $\chi(u; \bar{\xi}, s) = E_{\bar{\xi}}(X(\bar{\xi}, s)^u)$, then as $n \to \infty$,

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BPRE: $d(\bar{\xi}, s) = 0$ ($\Rightarrow \mathbb{E}|\log m(\xi_0)| = \infty$)

- θ is the shift operator, for any $\bar{\xi} = \{\xi_0, \xi_1, \cdots\}$, $\theta\bar{\xi} := \{\xi_1, \xi_2, \cdots\}$;
- $d(\bar{\xi}, s) := \lim_{n\to\infty} \frac{h_{n+1}(\bar{\xi},s)}{h_n(\theta\bar{\xi},s)} = \lim_{n\to\infty} \frac{h_{\xi_n}(\cdots(h_{\xi_0}(s))\cdots)}{h_{\xi_n}(\cdots(h_{\xi_1}(s))\cdots)}$.
- **Assumption**
  (A1) $\eta(\xi_0^{(0)} = 0) = 1$.
  (A2) $\eta(D) = 1$, where $D = \{\bar{\xi} : \text{for any } 0 < s < \infty,\ d(\bar{\xi}, s) = 0\}$.

Remark

(1) **(A1)**$\Longrightarrow$ $q(\bar{\xi}) = 0$.

(2) *(Tanny, '78)* $\mathbb{E}|\log m(\xi_0)| < \infty \Longrightarrow 0 < d(\bar{\xi}, s) \leqslant 1$ *w.p.1.*

(3) *conjecture ?* $\mathbb{E}|\log m(\xi_0)| = \infty \Longleftrightarrow$ $d(\bar{\xi}, s) = 0$ *w.p.1*

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(1) $W_n(\bar{\xi}, s) \longrightarrow W(\bar{\xi}, s)$, η-a.s..

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(1) Let $c_n(\bar{\xi})$ be a sequence of positive constants, such that $Z_n(\bar{\xi})/c_n(\bar{\xi})$ converges in distribution, and let $F_{\bar{\xi}}$ denote the distribution function of the limit. Then there are four cases:

$$(a)\ F_{\bar{\xi}}(0) = 1 \Longrightarrow \lim_n h_n(\bar{\xi}, s) c_n(\bar{\xi}) = \infty \quad \textit{for all} \quad 0 < s < \infty;$$

$$(b)\ F_{\bar{\xi}}(0) = F_{\bar{\xi}}(\infty) = 0 \Longrightarrow \lim_n h_n(\bar{\xi}, s) c_n(\bar{\xi}) = 0 \quad \textit{for all} \quad 0 < s < \infty;$$

$$(c)\ 1 > F_{\bar{\xi}}(0) = F_{\bar{\xi}}(\infty) > 0 \Longrightarrow \lim_n h_n(\bar{\xi}, t) c_n(\bar{\xi}) = \begin{cases} 0 & \textit{if} \ \ 0 < t < s_r \\ \infty & \textit{if} \ \ s_r < t < \infty; \end{cases}$$

$$(d)\ F_{\bar{\xi}}(0) < F_{\bar{\xi}}(\infty) \Longrightarrow \lim_n h_n(\bar{\xi}, s_i) c_n(\bar{\xi}) = 1.$$

2 Normalization by a sequence of functions

- For $0 \leqslant x < \infty$, let

$$y_n(\bar{\xi}, x) = f_{\xi_0}(\cdots(f_{\xi_{n-1}}(e^{-\frac{1}{x}}))\cdots), \ (y_n(\bar{\xi}, 0) = f_n(\bar{\xi}, 0)).$$

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- Let $U(\bar{\xi}) : [0, \infty) \to [0, \infty)$ be an increasing slowly varying function with $U(\bar{\xi}, 0) = 0$, $\lim_{x\to\infty} U(\bar{\xi}, x) = \infty$, and $\{c_n(\bar{\xi})\}$ a sequence of positive constants. We assume that $\bar{\xi}$ satisfies for any $s \in (0, \infty)$, $d(\bar{\xi}, s) = 0$.

Theorem (Hong & Z, 2019)

(1) *If*

$$H(\bar{\xi}, s) := \lim_n \left(U(\bar{\xi}, 1/h_n(\bar{\xi}, s))/c_n(\bar{\xi}) \right) \tag{2}$$

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Theorem (Hong & Z, 2019, Cont.)

(2) *On the other hand, if* $\{U(\bar{\xi}, Z_n(\bar{\xi}))/c_n(\bar{\xi})\}$ *converges in distribution to a distribution function* $F_{\bar{\xi}}$*, and define*

$$G(\bar{\xi}, x) = \inf\{y \mid 0 \leqslant y < \infty \text{ and } F_{\bar{\xi}}(y) \geqslant x\}, \quad 0 \leqslant x < \infty. \tag{3}$$

Then

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Theorem (Hong & Z, Cont.)

(3) *Under the condition of (2) and some other conditions*

$$\lim_n \frac{c_{n-1}(\theta\bar{\xi})}{c_n(\bar{\xi})} = \alpha(\bar{\xi}) > 0 \qquad (5)$$

exists, and

$$G(\bar{\xi}, e^{-s})/G(\theta\bar{\xi}, e^{-h_{\xi_0}(s)}) = \alpha(\bar{\xi}) \qquad \text{for } s \in (0, \infty). \qquad (6)$$

Furthermore, the distribution function $F_{\bar{\xi}}$ *and* $F_{\theta\bar{\xi}}$ *satisfy the functional equation*

$$F_{\bar{\xi}}(\alpha(\bar{\xi})u) = f_{\xi_0}(F_{\theta\bar{\xi}}(u)), \quad 0 \leqslant u < \infty, \qquad (7)$$

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Thank you for your attention!